Exponential and Logarithmic Functions

Using their properties

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This tutorial will focus on the properties of exponential and logarithmic functions.

Simplification and manipulation of functions, without the use of a calculator, will be investigated.
Let’s start with exponential functions.

Definition: An exponential function is in the form $f(x) = a^x$

- $a > 0$
- $a \neq 1$
- $a$ is the base
Here are some examples.

\[ f(x) = 3^x \quad f(x) = 5.1^x \]

\[ f(x) = \left(\frac{1}{3}\right)^x \quad f(x) = e^{x+5} \]

*Note: The base is a constant – the exponent is the variable*
What can we do with exponents?

What are their properties?
Basic Properties

- **division** – *subtract the exponents*
- **multiplication** – *add the exponents*
- **exponentiation** – *multiply the exponents*

In general, the bases must be the same in order to do any simplification.
Addition and Subtraction

We can only add up the number of each like term. The bases AND exponents must be exactly the same.

\[ f(x) = 3^x + 3^x \]

\[ f(x) = 2 \cdot 3^x \text{ or } 2 \left( 3^x \right) \]

the 2 represents how many you have

Common error: do NOT multiply the 2 and the 3
Multiplication

Add the exponents of terms with the same base.

\[ f(x) = (3^x)(3^4)(3^x) \]

\[ f(x) = 3^{2x+4} \]

Common error: do NOT multiply the 3’s
Division

Subtract the exponents of the terms with the same base.

\[ f(x) = \frac{2^x}{2^4} = 2^{x-4} \]

Common error: dividing the bases
Negative Exponents

\[ \frac{1}{2^x} = 2^{-x} \]

remember

Common error: making the base negative
Exponentiation

multiply the exponents

\[ f(x) = \left( 2^x \right)^4 \]

this means

\[ (2^x)(2^x)(2^x)(2^x) = 2^{x+x+x+x} \]

so

\[ f(x) = \left( 2^x \right)^4 = 2^{4x} \]
Summary

\[ a^x + a^x + a^x = 3a^x \]

\[ (a^x)(a^y)(a^z) = a^{x+y+z} \]

\[ \frac{a^x}{a^y} = a^{x-y} \]

\[ (a^x)^y = a^{xy} \]
OK, let’s try some problems.

Do the problem on your worksheet. Then click the mouse button to display the answer.

1. \(4^x + 3^x + 4^x = 2\left(4^x\right) + 3^x\)

2. \(5^x + 3^x + 5^x + 3^x + 3^w = 2\left(5^x\right) + 2\left(3^x\right) + 3^w\)

\(3^w\) has no like term

3. \(4^x + 4^4 + 3^x = 4^x + 4^4 + 3^x\)

there are no like terms
4. \((e^x)(e^{2x}) = e^{x+2x} = e^{3x}\)

*Just add the exponents.*

5. \(2^{5x+1} \cdot 2^{3-2x} = 2^{5x+1+3-2x} = 2^{3x+4}\)

6. \(4^x \cdot 2^{5x+1} = \left(2^2\right)^x \cdot 2^{5x+1} = 2^{2x} \cdot 2^{5x+1}\)

*The common base is 2, so change the 4 to 2 squared.*

*Now you have everything with the same base.*

\[2^{2x} \cdot 2^{5x+1} = 2^{2x+5x+1} = 2^{7x+1}\]
Now let’s talk about logarithms.

Definition: A logarithmic function is in the form $f(x) = \log_a x$

• $a > 0$

• $a \neq 1$

• $a$ is the base

• $x > 0$
Exponential functions and logarithmic functions are inverses of each other.

function \[ y = a^x \]

inverse \[ x = a^y \]

To solve for y, a new function, log is created.

\[ y = \log_a x \]
The output of a log is the exponent of the base, so

**a logarithm is an exponent**

**function**

\[ y = a^x \]

*base is a, exponent is x*

**inverse**

\[ x = a^y \]

\[ y = \log_a x \]

these are equivalent

*base is a, exponent is y*
Let’s look at some examples.

**function**

\[ y = 2^x \]

if \( x=2 \), what is \( y \)?

2 to the second power is 4, so \( y = 4 \)

(2, 4) would be a point that belongs to this function

**inverse**

\[ y = \log_2 x \quad \text{or} \quad x = 2^y \]

if \( x=4 \), what is \( y \)?

\( y \) is the exponent of 2 that results in 4, so \( y = 2 \)

(4, 2) would be a point that belongs to this function
let the base be 2

\[ y = \log_2 x \]

if \( x = 16 \), what is \( y \)?

\[ y = \log_2 16 \]

\[ 2^y = 16 \quad 2^4 = 16 \quad y = 4 \]

you can also re-write \( x \) in terms of the base to solve for \( y \)

\[ y = \log_2 16 = \log_2 2^4 \]

\[ y = 4 \]
\[ y = \log_4 \frac{1}{4} \]

we need to write \( \frac{1}{4} \) as a power of 4

\[ \frac{1}{4} = 4^{-1} \]

re-write our problem as

\[ y = \log_4 4^{-1} \]

\[ y = -1 \]
OK, let’s look at some problems

Do the problem on your worksheet. Then click the mouse button to display the answer.

1. \( y = \log_2 32 = 5 \)
   
   \( y = \log_2 32 = \log_2 2^5 \)
   
   \( y = 5 \)

2. \( y = \log_3 27 = 3 \)
   
   \( y = \log_3 27 = \log_3 3^3 \)
   
   \( y = 3 \)
3. \( y = \log_2 \frac{1}{8} = -3 \)

We need to write \( \frac{1}{8} \) as a power of 2

\[
\frac{1}{8} = (8)^{-1} = \left(2^3\right)^{-1} = 2^{-3}
\]

\[ y = \log_2 \frac{1}{8} = \log_2 2^{-3} \]

\[ y = -3 \]

4. \( y = \log_{\frac{1}{5}} \frac{1}{25} \)

\[
y = \log_{\frac{1}{5}} \frac{1}{25} = \log_{\frac{1}{5}} \left(\frac{1}{5}\right)^2
\]

\[ y = 2 \]
What can we do with logarithms?

What are their properties?
Basic Properties

There are three basic properties.

\[
\log_a (xy) = \log_a x + \log_a y
\]

\[
\log_a \frac{x}{y} = \log_a x - \log_a y
\]

\[
\log_a x^n = n \log_a x
\]

Let’s look at each one.
Product Rule

\[ \log_a (xy) = \log_a x + \log_a y \]

Remember, logs are exponents. When you multiply terms with the same base, you **ADD** the exponents.

Therefore, you can change the log of a product to the sum of the logs.

You must only have the log of a product to use this property.
Let’s look at an example to see why this is true.

\[ \log_2(4 \cdot 8) = \log_2(32) = \log_2 2^5 = 5 \]

but looking at this a different way,

\[ 4 \cdot 8 = \left(2^2\right) \left(2^3\right) = 2^{2+3} \]

\[ \log_2 4 \quad \log_2 8 \]

\[ \log_2(4 \cdot 8) = \log_2 4 + \log_2 8 = 2 + 3 = 5 \]
Quotient Rule

\[ \log_a \frac{x}{y} = \log_a x - \log_a y \]

When you divide terms with the same base, you **subtract** the exponents.

Therefore, you can change the log of a quotient to the **difference of logs**.

**You must have the log of a quotient to use this property.**
Exponent Rule

\[ \log_a x^n = n \log_a x \]

When you have a term with an exponent that is raised to another power, you \textbf{MULTIPLY} the exponents.

You must have the \textbf{log of a term and the term is to a power} to use this property.
Let’s look at an example to see why this is true.

\[
\log_2 4^3 = \log_2 64 = \log_2 2^6 = 6
\]

but looking at this a different way,

\[
4^3 = (4)(4)(4) = (2^2)(2^2)(2^2) = 2^{2+2+2}
\]

there are 3 of the log 4’s, so find one of them and multiply by 3.

\[
\log_2 4^3 = 3(\log_2 4) = 3(\log_2 2^2) = 3(2) = 6
\]
Why are these properties important?

• changes multiplication to addition
• changes division to subtraction
• changes exponentiation to a product
What are some common errors?

**Multiplication:**

\[ \log_a (xy) \quad \text{the log of a product} \]
\[ = \log_a x + \log_a y \]
\[ (\log_a x)(\log_a y) \quad \text{the product of logs} \]

The first can be changed to a sum.

The second cannot be changed.
**Division:**

\[
\log_a \frac{x}{y} \quad \text{the log of a quotient}
\]

\[
= \log_a x - \log_a y
\]

\[
\frac{\log_a x}{\log_a y} \quad \text{the quotient of two logs}
\]

*The first can be changed to a difference.*

*The second cannot be changed.*
**Exponentiation:**

\[ \log_a x^n \]

*the log of a term to a power*

*the x is to the nth power, not the log*

\[ = n \log_a x \]

\[ (\log_a x)^n \]

*the entire log is to the nth power*

*The first can be changed to a product.*

*The second **cannot** be changed.*
Addition/Subtraction:

\[ \log_a (x + y) \]

The first is the log of a sum - must have the sum in parentheses.

The common error is to use the distributive property and “multiply” the log through the sum.

This cannot be changed.

\[ \log_a x + \log_a y = \log_a xy \]

The second is the sum of logs. It can be changed to the log of a product.
The natural logarithm has a base of “e”. Since it is used frequently, there is an abbreviated notation for it.

\[
\begin{align*}
\log_e x & \quad \text{these are equivalent} \\
\ln x &
\end{align*}
\]
The properties for the natural log are the same as any other log, but we use the abbreviated notation.

\[
\ln(xy) = \ln x + \ln y
\]

\[
\frac{x}{y} = \ln x - \ln y
\]

\[
\ln x^n = n \ln x
\]
Let’s look at some examples.

Change from a product to a sum.

\[
\ln(x^2y) = \ln x^2 + \ln y = 2 \ln x + \ln y
\]

use the product rule
use the exponent rule

You can also change from a sum to a product.

\[
\ln 2x + \ln x = \ln (2x \cdot x) = \ln 2x^2
\]

use the product rule

Do not combine logs with different bases!

\[
\ln x + \log_3 x
\]

nothing changes here
\[
\log_5 \frac{3}{2x} = \log_5 3 - \log_5 2x
\]

*use the quotient rule*

\[
= \log_5 3 - (\log_5 2 + \log_5 x)
\]

*use the product rule*

*make sure to use parentheses*

\[
= \log_5 3 - \log_5 2 - \log_5 x
\]
\[
\ln \sqrt{\frac{3x^3}{x^2 - 3}} = \ln \left( \frac{3x^3}{x^2 - 3} \right)^{1/2} = \frac{1}{2} \ln \left( \frac{3x^3}{x^2 - 3} \right)
\]

- **use exponent for square root**
- **exponent rule**
- \[= \frac{1}{2} \left( \ln 3x^3 - \ln(x^2 - 3) \right) \]
- **quotient rule**
- \[= \frac{1}{2} \left( \ln 3 + \ln x^3 - \ln(x^2 - 3) \right) \]
- **product rule**
- \[= \frac{1}{2} \left( \ln 3 + 3 \ln x - \ln(x^2 - 3) \right) \]
- **exponent rule**
- \[= \frac{1}{2} \ln 3 + \frac{3}{2} \ln x - \frac{1}{2} \ln(x^2 - 3) \]
- **multiply**
Some special cases

\[ 4^0 = 1 \quad e^0 = 1 \quad 7.6^0 = 1 \]

Any real number (except 0) to the 0 power is 1.

Thus,

\[ \log_4 1 = \log_4 4^0 = 0 \]
\[ \ln 1 = 0 \]
\[ \log_{7.6} 1 = 0 \]

The log of 1 is zero, no matter what the base.
Some special cases

$4^1 = 4 \quad e^1 = e \quad 7.6^1 = 7.6$

thus,

$\log_4 4 = \log_4 4^1 = 1$

$\ln e = 1$

$\log_{7.6} 7.6 = 1$

so $\log_a a = 1$
The log of zero is undefined for all bases!

$$\log_a 0 = \text{undefined}$$

$$\ln 0 = \text{undefined}$$
OK, let’s look at some problems

Do the problem on your worksheet. Then click the mouse button to display the answer.

Write the following as a sum.
(Some cannot be re-written.)

5. \( y = \log_2 (4x) = \log_2 4 + \log_2 x \)

6. \( y = \log_3 2x^2 = \log_3 2 + \log_3 x^2 \)
   \[= \log_3 2 + 2\log_3 x\]

7. \( y = \ln \frac{\sqrt{x}}{x-3} = \ln x^{1/2} - \ln (x-3) \)
   \[= \frac{1}{2} \ln x - \ln (x-3)\]
8. \( y = \ln(x + 3) \ln 3x \)

\[ = \ln(x + 3) \left( \ln 3 + \ln x \right) \]
You should now be able to use the properties of exponential and logarithmic functions.

If you would like more practice, there are additional worksheets available.